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Exact solutions of relativistic perfect fluid hydrodynamics for a QCD Equation of State

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Abstract. We generalize a previously known class of exact analytic solutions of relativistic perfect fluid hydrodynamics for the first time to arbitrary temperature-dependent Equation of State. We investigate special cases of this class of solutions, in particular, we present hydrodynamical solutions with an Equation of State determined from lattice QCD calculations. We discuss the phenomenological relevance of these solutions as well.

1 Introduction

The interest in relativistic hydrodynamics has grown in the past years mainly due to the discovery of the almost perfect fluidity of the experimentally created Quark-Gluon Plasma at the Relativistic Heavy Ion Collider (RHIC) [1,2]. Hydrodynamical models aim to describe the space-time picture of heavy-ion collisions and infer the relation between experimental observables and the initial conditions. Besides numerical simulations there is also interest in models where exact solutions of the hydrodynamical equations are used. Aside from historical exact solutions (such as the Landau-Khalatnikov solution [3,4] and the Hwa-Bjorken solution [5,6]) which were important in the development of hydrodynamical model building in high-energy physics, one can find recent examples of exact solutions which yield analytic insight into the dynamics of heavy-ion collisions (see, *e.g.*, refs. [7–9] and references therein).

In this paper we generalize a previously known class of exact solutions of relativistic perfect fluid hydrodynamics [10] to the case of arbitrary, temperature-dependent speed of sound, as detailed in the next sections. The mentioned class of solutions form the basis of the relativistic Buda-Lund hydrodynamical model [11]. This model yields a successful description of hadronic observables at RHIC energies (such as the pseudorapidity and transverse momentum dependence of the azimuthal anisotropy of different hadrons as well as the HBT radii [11]), and the reconstructed final state in this model corresponds to simple explicit scaling solutions of hydrodynamics. The same final state however can be reached from many different initial states, depending on the Equation of State [12]. If one is given a temperature dependent speed of sound as Equation of State, the solution presented in this paper thus can be used to determine the initial state from the reconstructed final state of a heavy-ion collision assuming the validity of perfect fluid hydrodynamics. As an example, we describe the time dependence of the system if one assumes an Equation of State calculated in lattice QCD.

The solutions given in this paper are the first exact analytic solutions of 1+3 dimensional relativistic hydrodynamics, to utilize an arbitrary Equation of State[1].

2 Basic equations

Let us adopt the following notational conventions: the space-time coordinates shall be $x^\mu = (t, \mathbf{r})$, where $\mathbf{r} = (r_x, r_y, r_z)$ is the spatial coordinate and t is the time in lab-frame. The metric tensor is $g_{\mu\nu} = \mathrm{diag}\,(1, -1, -1, -1)$. (We denote space-time indices by Greek letters, space indices by Latin letters and assume the summation convention.) The fluid four-velocity is $u^\mu = \gamma\,(1, \mathbf{v})$, where $\mathbf{v}$ is the three-velocity with $v = |\mathbf{v}|$, and $\gamma = 1/\sqrt{1-v^2}$. The thermodynamical quantities are denoted as follows: p is the pressure, ε is the energy density, σ is the entropy density, T is the temperature. If the fluid consists of individual conserved particles, or if there is some conserved charge, then the conserved number density is denoted by n, and the corresponding chemical potential by μ. (For more than one conserved number densities, one may use indices to

[1] Note that it has been discussed, in ref. [13], that the entropy flow can be calculated with an arbitrary EoS (Equation of State, speed of sound) from the Khalatnikov-potential, once the solution of the general Khalatnikov equation is known.

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distinguish them.) All these quantities have dependence on x^μ, but mostly this will not be explicitly written.

The basic hydrodynamical equations are the continuity and energy-momentum–conservation equations

$$\partial_\mu (n u^\mu) = 0, \quad (1)$$
$$\partial_\nu T^{\mu\nu} = 0. \quad (2)$$

The energy-momentum tensor of a perfect fluid is

$$T^{\mu\nu} = (\varepsilon + p) u^\mu u^\nu - p g^{\mu\nu}. \quad (3)$$

Equation (2) can be then transformed into (by projecting it orthogonal and parallel to u^μ, respectively)

$$(\varepsilon + p) u^\nu \partial_\nu u^\mu = (g^{\mu\nu} - u^\mu u^\nu) \partial_\nu p, \quad (4)$$
$$(\varepsilon + p) \partial_\nu u^\nu + u^\nu \partial_\nu \varepsilon = 0. \quad (5)$$

Equation (4) is the relativistic Euler equation, while eq. (5) is the relativistic form of the energy conservation equation. In appendix A we recall the well-known fact that eq. (5) is equivalent to the entropy conservation equation

$$\partial_\mu (\sigma u^\mu) = 0. \quad (6)$$

An analytic hydrodynamical solution is a functional form of ε, p, T, u^μ (and, if dealt with, n), which solves eqs. (4) and (5), and, if present, n also solves eq. (1). The quantities ε, p, T, and also σ and n are subject to the Equation of State (EoS), which closes the set of equations. We investigate the following EoS:

$$\varepsilon = \kappa(T) p, \quad (7)$$

while the speed of sound c_s is calculated as

$$c_s = \sqrt{\frac{\partial p}{\partial \varepsilon}}, \quad (8)$$

i.e. for constant κ, $c_s = 1/\sqrt{\kappa}$. We see from this that the case of temperature-dependent c_s is equivalent to the case of a temperature-dependent κ coefficient. For the case when there is a conserved n number density, we also use the well-known relation for ideal gases,

$$p = nT. \quad (9)$$

For the case of $\kappa(T) =$ constant, an ellipsoidally symmetric solution of the hydrodynamical equations is presented in ref. [10],

$$u^\mu = \frac{x^\mu}{\tau}, \quad \tau = \sqrt{t^2 - r^2} = \sqrt{x_\mu x^\mu}, \quad (10)$$

$$n = n_0 \frac{V_0}{V} \nu(s), \quad T = T_0 \left(\frac{V_0}{V}\right)^{\frac{1}{\kappa}} \frac{1}{\nu(s)}, \quad V = \tau^3, \quad (11)$$

where n_0 and T_0 correspond to the proper time when the arbitrarily chosen volume V_0 was reached (*i.e.* $\tau_0 = V_0^{1/3}$), $\nu(s)$ is an arbitrary function of s, which is defined as

$$s = \frac{r_x^2}{X^2} + \frac{r_y^2}{Y^2} + \frac{r_z^2}{Z^2}, \quad (12)$$

where X, Y, and Z are the time (lab-frame time t) dependent principal axes of an expanding ellipsoid. They have the explicit time dependence as

$$X = \dot{X}_0 t, \quad Y = \dot{Y}_0 t, \quad Z = \dot{Z}_0 t, \quad (13)$$

with $\dot{X}_0$, $\dot{Y}_0$, $\dot{Z}_0$ constants. The quantity s has ellipsoidal level surfaces, and obeys $u^\nu \partial_\nu s = 0$. We call s a *scaling variable*, and V the effective volume of a characteristic ellipsoid.

Note that with the $X(t)$, $Y(t)$, $Z(t)$ time-dependent axes introduced as above, we can write the velocity field in the form of

$$\mathbf{v} = \left(\frac{\dot{X}}{X} r_x, \frac{\dot{Y}}{Y} r_y, \frac{\dot{Z}}{Z} r_z\right), \quad (14)$$

which underlines the resemblance of this solution to certain non-relativistic exact solutions with Hubble-like expansion [14, 15]. This solution is *non-accelerating*, *i.e.* obeys $u^\nu \partial_\nu u^\mu = 0$. In the next section we present a generalization of this class of solutions to more general EoS. The new solutions will be presented in sect. 4, while sect. 3 details their derivation.

3 General Equation of State

In order to find more general solutions, where a temperature-dependent EoS can be used (as in eq. (7)), for a given u^μ velocity field we may *define* the V and s quantities by their properties that

$$u^\mu \partial_\mu V = V \partial_\mu u^\mu, \quad u^\mu \partial_\mu s = 0. \quad (15)$$

With these quantities, eq. (1) is automatically solved (for the case when there is a conserved charge present) if

$$n = n_0 \frac{V_0}{V} \nu(s) \quad (16)$$

again, with arbitrary $\nu(s)$ function. Similarly, eq. (6) is also automatically solved if

$$\sigma = \sigma_0 \frac{V_0}{V} \nu(s). \quad (17)$$

To solve the (5) energy equation, we must make a distinction between the two possible cases. The first case is if we take a conserved n into account, and use the EoS $\varepsilon = \kappa(T) p$, $p = nT$ as in eqs. (7) and (9). The second case is when we do not consider any conserved n. In appendix B we show that in both of these two cases the energy equation eq. (5) can be transformed to an equation for T: in the first case with conserved n, we have

$$u^\mu \left[\frac{\partial_\mu V}{V} + \frac{\mathrm{d}(\kappa T)}{\mathrm{d}T} \frac{\partial_\mu T}{T}\right] = 0, \quad (18)$$

while, in the case where there is no conserved n, we have

$$u^\mu \left[\frac{\partial_\mu V}{V} + \left(\frac{1}{\kappa + 1} \frac{\mathrm{d}\kappa}{\mathrm{d}T} + \frac{\kappa}{T}\right) \partial_\mu T\right] = 0. \quad (19)$$

Remarkably, these equations are not the same (however, we may note that in the case when $\kappa = \text{const.}$, they yield the same condition). We call eqs. (18) and (19) the temperature equations for the two cases. With the introduction of the $f(T)$ function as

$$f(T) = \exp\left\{\int_{T_0}^{T}\left(\frac{1}{\beta}\frac{\mathrm{d}}{\mathrm{d}\beta}\left[\kappa(\beta)\beta\right]\right)\mathrm{d}\beta\right\}, \qquad (20)$$

for the case of conserved n; while for the case without a conserved n, as

$$f(T) = \exp\left\{\int_{T_0}^{T}\left(\frac{\kappa(\beta)}{\beta} + \frac{1}{\kappa(\beta)+1}\frac{\mathrm{d}\kappa(\beta)}{\mathrm{d}\beta}\right)\mathrm{d}\beta\right\}, \qquad (21)$$

the temperature equations can be cast in the following universal form:

$$u^\mu\left[\frac{\partial_\mu V}{V} + \frac{\partial_\mu f(T)}{f(T)}\right] = 0. \qquad (22)$$

For any given $\kappa(T)$ function we can then determine $f(T)$. The solution of the above equation is then

$$f(T) = \frac{V_0}{V}\xi(s) \;\Rightarrow\; T = f^{-1}\left(\frac{V_0}{V}\xi(s)\right), \qquad (23)$$

with an arbitrary $\xi(s)$ function. (For convenience, we may normalize $\xi(s)$ so that $\xi(0) = 1$.) Taking into account the $u^\mu\partial_\mu s = 0$ relation, it is easy to see that eq. (23) indeed solves eq. (22).

An important point is that if $\kappa = \text{constant}$, then eqs. (20), (21) and (23) simplify to

$$f(T) = \left(\frac{T}{T_0}\right)^\kappa \;\Rightarrow\; T = T_0\left(\frac{V_0}{V}\right)^{1/\kappa}\xi(s)^{1/\kappa}, \qquad (24)$$

so we indeed get back the original solution of ref. [10].

As a generalization of the solution recalled in the previous section, we assume that u^μ and thus s and V have the same forms as in eqs. (10) and (12),

$$u^\mu = \frac{x^\mu}{\tau}, \quad V = \tau^3, \quad s = \frac{r_x^2}{\dot{X}_0^2t^2} + \frac{r_y^2}{\dot{Y}_0^2t^2} + \frac{r_z^2}{\dot{Z}_0^2t^2}. \qquad (25)$$

Now let us check the remaining equation, the Euler equation of (4). For this velocity field, $u^\nu\partial_\nu u^\mu = 0$, the Euler equation is equivalent to

$$\partial_\mu p = u_\mu u^\nu\partial_\nu p. \qquad (26)$$

In the case of vanishing n, using the thermodynamic relation $\mathrm{d}p = \sigma\mathrm{d}T$, eq. (26) simplifies to

$$\partial_\mu T = u_\mu u^\nu\partial_\nu T. \qquad (27)$$

Let us substitute the expression of T from eq. (23), and consider that $\partial_\mu s \neq 0$, and f^{-1} cannot be constant. We find that eq. (27) is equivalent (for any $\kappa(T)$, thus for any $f(T)$ function) to

$$f^{-1\prime}\left(\frac{V_0}{V}\xi(s)\right)\frac{\xi'(s)}{\xi(s)}\partial_\mu s = 0 \quad\Rightarrow\quad \xi(s) = \text{const.} \qquad (28)$$

In the case of non-vanishing n, using eq. (16) and $p = nT$, the Euler equation for our non-accelerating velocity field transforms into the following equation:

$$T\partial_\mu n + n\partial_\mu T = Tu_\mu u^\nu\partial_\nu n + nu_\mu u^\nu\partial_\nu T. \qquad (29)$$

Substituting n and T from eqs. (16) and (23), and the definition of V, we get, from this equation, the following constraint:

$$\left[\frac{\nu'(s)}{\nu(s)} + \varphi\left(\frac{V_0}{V}\xi(s)\right)\frac{\xi'(s)}{\xi(s)}\right]\partial_\mu s = 0, \qquad (30)$$

where we have introduced the following function,

$$\varphi(y) = \frac{yf^{-1\prime}(y)}{f^{-1}(y)}. \qquad (31)$$

Since $\partial_\mu s \neq 0$, we see from eq. (30) that there are two simple cases: for any EoS (*i.e.* for any $\kappa(T)$ and thus any φ function) we get a solution if $\nu(s) = \xi(s) = 1$. The other possibility is if $\kappa = \text{const.}$ It is easy to see that in this case

$$\varphi(y) = \frac{1}{\kappa} = \text{const.}, \qquad (32)$$

and so eq. (30) is solved if $\xi = \nu^{-1/\kappa}$ and so from eq. (24) we get $T = T_0(V_0/V)^{1/\kappa}\nu^{-1}(s)$, *i.e.* the same as in eq. (11) (see appendix C, for details). In this case we indeed obtain the known solution of ref. [10], recited in eqs. (10)–(12).

4 New solutions for general Equation of State

Summarizing and rewriting the results presented in the previous section, we found new solutions to the relativistic hydrodynamical equations for arbitary $\varepsilon = \kappa(T)p$ Equation of State, and these are the first solutions of their kind (*i.e.* with a non-constant EoS). In the case where we do not consider any conserved n density, the solution can be presented in the following form, in terms of u^μ, σ and T, with T given in an implicit form:

$$\sigma = \sigma_0\frac{\tau_0^3}{\tau^3}, \qquad (33)$$

$$u^\mu = \frac{x^\mu}{\tau}, \qquad (34)$$

$$\frac{\tau_0^3}{\tau^3} = \exp\left\{\int_{T_0}^{T}\left(\frac{\kappa(\beta)}{\beta} + \frac{1}{\kappa(\beta)+1}\frac{\mathrm{d}\kappa(\beta)}{\mathrm{d}\beta}\right)\mathrm{d}\beta\right\}, \qquad (35)$$

where eq. (33) was obtained similarly to eq. (11), while eq. (35) can be derived from eqs. (21) and (23). Also, for the case when the pressure is expressed as $p = nT$ with some conserved n density, the new solution is written in terms of u^μ, T and n as

$$n = n_0 \frac{\tau_0^3}{\tau^3}, \tag{36}$$

$$u^\mu = \frac{x^\mu}{\tau}, \tag{37}$$

$$\frac{\tau_0^3}{\tau^3} = \exp\left\{\int_{T_0}^{T} \left(\frac{1}{\beta}\frac{\mathrm{d}}{\mathrm{d}\beta}\left[\kappa(\beta)\beta\right]\right)\mathrm{d}\beta\right\}, \tag{38}$$

where again, the last equation was obtained from eqs. (20) and (23). Note that these solutions form simple generalization of the $\nu(s) = 1$ case of the solutions of ref. [10], and the latter also represents a relativistic generalization of the solution presented in ref. [15].

Note that in both cases, the quantities denoted by the subscript 0 (n_0, T_0, σ_0) correspond to the proper-time τ_0, which can be chosen arbitrarily. If, for example, τ_0 is taken to be the freeze-out proper time, then T_0 is the freeze-out temperature.

An important point is that in the case when $p = nT$ and n is conserved, for some choices of the $\kappa(T)$ function our solution becomes ill-defined. The criterion of

$$\frac{\mathrm{d}}{\mathrm{d}T}\left(\kappa(T)\,T\right) > 0 \tag{39}$$

limits the applicability of solutions for the case of conserved n presented here. In the case when for some T range $\frac{\mathrm{d}}{\mathrm{d}T}(\kappa(T)\,T)$ becomes negative, the implicit form of eq. (38) cannot be inverted to give a unique $T(\tau)$ function. Such domains of T indeed might exist in some parameterizations of a lattice QCD Equation of State around the quark-hadron transition temperature (as detailed in the next section, in particularly on fig. 1). This is due to the fact that the condition $\frac{\mathrm{d}}{\mathrm{d}T}(\kappa(T)\,T) > 0$ is equivalent to $\partial T(\varepsilon, n)/\partial\varepsilon > 0$ for constant n, which is again equivalent to the positiveness of the specific heat, which may not be valid at phase transitions. However, even for these cases, one can use the solution without conserved n, presented in eqs. (33)–(35). This is the physically relevant solution in this case, since at the transition temperature a conserved density n yielding pressure as $p = nT$ may not be present.

Let us briefly mention another possibility, when κ is a function of the pressure p and not that of the temperature T. In this case a new solution, similarly to the previous ones is the following:

$$\sigma = \sigma_0 \frac{\tau_0^3}{\tau^3}, \tag{40}$$

$$u^\mu = \frac{x^\mu}{\tau}, \tag{41}$$

$$\frac{\tau_0^3}{\tau^3} = \exp\left\{\int_{p_0}^{p} \left(\frac{\kappa(\beta)}{\beta} + \frac{\mathrm{d}\kappa(\beta)}{\mathrm{d}\beta}\right)\frac{\mathrm{d}\beta}{\kappa(\beta)+1}\right\}, \tag{42}$$

i.e. almost the same as in eq. (35), except that here the integration variable is the pressure p. If however the pressure can be written as a function of temperature, *i.e.* as $p(T)$, an integral transformation can be made with and we get back eq. (35), so in this case these solutions are identical. This solution may be used if a $\kappa(p)$ function is given (without relation to the temperature) by an arbitrary energy density function $\varepsilon(p) = \kappa(p)p$.

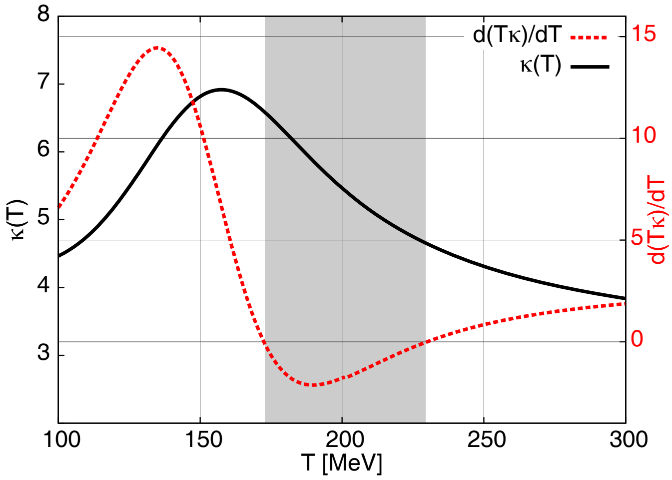

Fig. 1. (Colour on-line) The temperature dependence of the EoS parameter κ from ref. [16] is shown with the solid black curve. Note that in the shaded T range (173 MeV–230 MeV) $\frac{\mathrm{d}}{\mathrm{d}T}(\kappa(T)\,T)$ (red dashed line) becomes negative, thus the implicit form of eq. (38) cannot be inverted to give a unique $T(\tau)$ function. Hence we will substitute this $\kappa(T)$ in the hydrodynamic solution shown in eqs. (33)–(35).

5 Application

Recently, a QCD Equation of State has been calculated by the Budapest-Wuppertal group in ref. [16]. Here (in their eq. (3.1) and table 2) they give a parametrization of the trace anomaly as a function of temperature. Hence the pressure, the energy density and finallly the EoS parameter κ can be calculated, as a function of the temperature. We did this calculation, and got the $\kappa(T)$ function as shown in fig. 1. Note however, that in this calculation for some T range $\frac{\mathrm{d}}{\mathrm{d}T}(\kappa(T)\,T)$ becomes negative, as also shown in fig. 1. Hence the implicit form of eq. (38) cannot be inverted to give a unique $T(\tau)$ function. We can still use the solution without conserved number density n, presented in eqs. (33)–(35).

We utilized the obtained $\kappa(T)$ and calculated the time evolution of the temperature of the fireball from this solution of relativistic hydrodynamics. The result is shown in fig. 2. Clearly, temperature falls off almost as fast as in case of a constant $\kappa = 3$, an ideal relativistic gas. Hence a given freeze-out temperature yields a significantly higher initial temperature than a higher κ (*i.e.* a low c_s^2) would.

Let us fix the freeze-out temperature to be $T_f = 170\,\mathrm{MeV}$, for example (and let all the quantities with subscript 0 correspond to the freeze-out, thus index them with f). In this case, already at $0.3\times\tau_f$ (30% of the freeze-out time), temperatures 2.5–3× higher than at the freeze-out

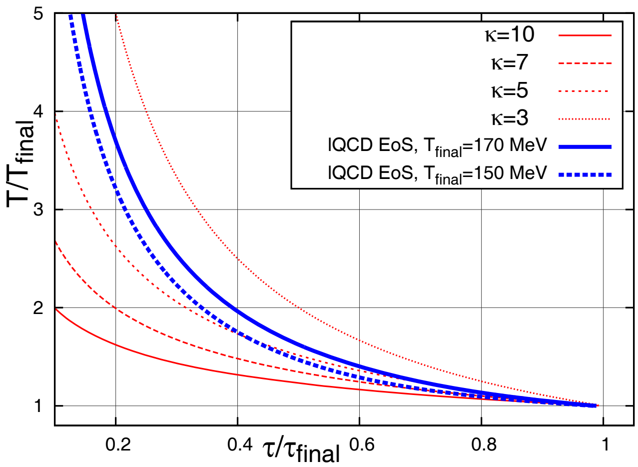

Fig. 2. (Colour on-line) Time dependence of the temperature $T(\tau)$ (normalized with the freeze-out time τ_f and the freeze-out temperature T_f). The four thin red lines show this dependence in case of constant κ values, while the thicker blue lines show results based on the EoS of ref. [16]. The resulting curve slightly depends on the value of T_f. It is clear, however, that the temperature fall-off is almost as fast in the lQCD EoS case as in the case of fixed $\kappa = 3$, which resembles a relativistic ideal gas. This means that a fixed freeze-out temperature (which cannot vary too much due to the known quark-hadron transition temperature) results in a very high initial temperature.

can be reached. To give a quantitative example, if

$$\tau_f = 8\,\text{fm}/c \quad \text{and} \quad \tau_{\text{init}} = 1.5\,\text{fm}/c, \tag{43}$$

then

$$T_f = 170\,\text{MeV} \;\Rightarrow\; T_{\text{init}} \approx 550\,\text{MeV} \tag{44}$$

(and even higher if τ_{init} is smaller). The QCD Equation of State of ref. [16] and this hydro solution yields a general $T(\tau)$ dependence. If the freeze-out temperature T_f and the time evolution duration $\tau_f/\tau_{\text{init}}$ are known, the initial temperature of the fireball can be easily calculated.

6 Conclusion

We have presented the first analytic solutions of the equations of relativistic perfect fluid hydrodynamics for general temperature-dependent speed of sound (*i.e.* general Equation of State). They can be seen as generalizations of previously known exact solutions [10]. However, our new solutions are spherically symmetric, thus possible generalizations of them are definitely worth exploring: solutions for the non-accelerating case and for more general ellipsoidal symmetry would be able to analytically explore the time evolution of other hadronic observables such as the elliptic flow (v_2).

We have shown how to use our solutions to fully utilize a lattice QCD Equation of State for exploring the initial state of heavy-ion reactions based on the reconstructed final state in the Buda-Lund hydrodynamical model. In $\sqrt{s_{NN}} = 200\,\text{GeV}$ Au+Au collisions, our investigations reveal a very high initial temperature consistent with calculations based on the measurement spectrum of low momentum direct photons [17]. If it is given a temperature-dependent direct photon emission function, then this model can be used to calculate direct photon spectra to be compared to measurements, as in ref. [18], but with a realistic Equation of State.

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Appendix A. Entropy conservation

The fundamental thermodynamical relations connecting ε, T, σ, p, and any types of n_a conserved charges (the index a, here, distinguishes is between chemical component types) and the corresponding μ_a chemical potentials are

$$\varepsilon + p = T\sigma + \sum_a n_a \mu_a, \tag{A.1}$$

$$\mathrm{d}\varepsilon = T\mathrm{d}\sigma + \sum_a \mu_a \mathrm{d}n_a, \tag{A.2}$$

$$\mathrm{d}p = \sigma\mathrm{d}T + \sum_a n_a \mathrm{d}\mu_a. \tag{A.3}$$

In the case where there are no conserved charges, similar relations hold with all n_a and μ_a variables omitted. Substituting these in the (5) energy conservation equation, we immediately obtain the continuity equation for the entropy density σ (simplified for the case of a single-component fluid)

$$T\sigma\partial_\mu u^\mu + Tu^\mu\partial_\mu\sigma + n\partial_\mu u^\mu + u^\mu\partial_\mu n = 0, \tag{A.4}$$

which is, for conserved (or vanishing) n, equivalent to

$$\partial_\nu\left(\sigma u^\nu\right) = 0, \tag{A.5}$$

which is the entropy conservation, eq. (6).

Appendix B. The temperature equations

In the case when there is no conserved n, we can substitute the (A.2) and (A.3) thermodynamic relations for vanishing n in eq. (5). Using the EoS as $\varepsilon = \kappa(T)p$, and $\varepsilon + p = T\sigma$ and $\mathrm{d}p = \sigma\mathrm{d}T$ we obtain, from eq. (5), the following:

$$T\sigma\left[\partial_\mu u^\mu + \frac{1}{\kappa+1}\frac{\mathrm{d}\kappa}{\mathrm{d}T}u^\mu\partial_\mu T\right] + \kappa\sigma\; u^\mu\partial_\mu T = 0, \tag{B.1}$$

which is, by using eq. (15) again, equivalent to eq. (19), as was to be demonstrated.

Next, we would like to obtain an equation for the temperature with our specific Equation of State as in eq. (7)

$(\varepsilon = \kappa(T)\,p)$, in the case when there is a conserved n and $p = nT$. We can substitute these into eq. (5), and use the (1) continuity equation for n to infer that eq. (5) is equivalent to the following:

$$T\partial_\mu u^\mu + \frac{\mathrm{d}}{\mathrm{d}T}(\kappa T)\,u^\mu\partial_\mu T = 0. \tag{B.2}$$

Introducing V by using eq. (15), we immediately see that this is equivalent to eq. (18), as it is stated in the text.

Let us also show how the solution for a given $\kappa(p)$, described in eqs. (40)–(42) can be obtained. In that case, instead of substituting the temerature to eq. (5), we write up the equation using the $\kappa(p)$ function and the relation $\varepsilon = \kappa \cdot p$, similarly to the previous cases,

$$u^\nu\left[\frac{\partial_\nu V}{V} + \left(\frac{\kappa}{p} + \frac{\mathrm{d}\kappa}{\mathrm{d}p}\right)\frac{\partial_\nu p}{\kappa+1}\right] = 0. \tag{B.3}$$

This equation is then solved by the implicit formula on the pressure, given in eq. (42)

Appendix C. Euler equation for a conserved charge

In the case of non-vanishing n, the Euler equation of eq. (29) can be expressed as

$$\frac{\partial_\mu n}{n} + \frac{\partial_\mu T}{T} = u_\mu\frac{u^\nu}{n}\partial_\nu n + u_\mu\frac{u^\nu}{T}\partial_\nu T. \tag{C.1}$$

Using eqs. (16) and (23), we find

$$\frac{\partial_\mu T}{T} = \frac{{f^{-1}}'\left(\frac{V_0}{V}\xi(s)\right)\frac{V_0}{V}\xi(s)}{f^{-1}\left(\frac{V_0}{V}\xi(s)\right)}\left[\frac{\xi'(s)}{\xi(s)}\partial_\mu s - \frac{\partial_\mu V}{V}\right], \tag{C.2}$$

$$\frac{\partial_\mu n}{n} = \frac{\nu'(s)}{\nu(s)}\partial_\mu s - \frac{\partial_\mu V}{V}. \tag{C.3}$$

Multiplying these by u^μ and substituting $\mu \to \nu$, we get

$$\frac{u^\nu}{T}\partial_\nu T = -\frac{{f^{-1}}'\left(\frac{V_0}{V}\xi(s)\right)\frac{V_0}{V}\xi(s)}{f^{-1}\left(\frac{V_0}{V}\xi(s)\right)}\partial_\nu u^\nu, \tag{C.4}$$

$$\frac{u^\nu}{n}\partial_\nu n = -\partial_\nu u^\nu. \tag{C.5}$$

In our case of eq. (25), however,

$$u^\mu\partial_\nu u^\nu = \frac{\partial_\mu V}{V}. \tag{C.6}$$

Substituting eqs. (C.2)–(C.6) to eq. (C.1), we get

$$\left[\frac{\nu'(s)}{\nu(s)} + \frac{{f^{-1}}'\left(\frac{V_0}{V}\xi(s)\right)\frac{V_0}{V}\xi(s)}{f^{-1}\left(\frac{V_0}{V}\xi(s)\right)}\frac{\xi'(s)}{\xi(s)}\right]\partial_\mu s = 0, \tag{C.7}$$

which is the same as eq. (30).

If $\kappa = \text{const.}$, $f(T)$ is expressed in eq. (24). From this, using the definition of $\varphi(y)$ in eq. (31),

$${f^{-1}}'(y) = \frac{T_0 y^{1/\kappa-1}}{\kappa} = \frac{f^{-1}(y)}{\kappa y} \Rightarrow \varphi(y) = \frac{1}{\kappa}, \tag{C.8}$$

thus eq. (C.7) transforms to

$$\left[\frac{\nu'(s)}{\nu(s)} + \frac{1}{\kappa}\frac{\xi'(s)}{\xi(s)}\right]\partial_\mu s = 0, \tag{C.9}$$

which is identically zero if $\xi = \nu^{-1/\kappa}$, thus, from eq. (24),

$$T = T_0\left(\frac{V_0}{V}\right)^{1/\kappa}\nu(s)^{-1}, \tag{C.10}$$

and we indeed obtain the known solution of ref. [10] shown in eqs. (10)–(12).

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