Classical and relativistical hydrodynamics

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Outline

1. Classical hydrodynamics
    - Conservation law, basic equation
    - An example solution
    - When the relativistic effects are interesting or important

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    - Some archaical (so called "well-known") solution from the past
    - An example solution with more details

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2. Relativistic hydrodynamics
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    - Some archaical (so called "well-known") solution from the past
    - An example solution with more details
3. More realistic EoS
    - What is the problem with $\kappa =$const. ?
    - What should or could we do?
    - Some lQCD results and their use

Classical hydrodynamics – Basic equation

- Continuity equation, Euler-equation, Energy-equation
- We can calculate these equation from Boltzman-equation
- Another way is the classical field theory
- We use the Lagrangian picture or coordinates
  $\rightarrow$ We can use the point particle approach
- The usual approach is the Euler coordinates. We have transformation rule:

$$v_L(\mathbf{r}_0, t) = v_E(\mathbf{r}(\mathbf{r}_0, t), t)$$

from this

$$\frac{dF_L}{dt} = \frac{\partial \mathbf{F}_E(\mathbf{r}, t)}{\partial t} + (\mathbf{v}\nabla)\mathbf{F}_E(\mathbf{r}, t)$$

Let introduce $\mathbf{J}$ Jacobi-matrix and its determinant:

$$J_{ab} = \frac{\partial u_a}{\partial r_b} \ , \ \ J = \det\mathbf{J}$$

- so we can calculate:

$$\rho d^3u = \rho_0 d^3r \ \rightarrow \ \rho = \frac{\rho_0(\mathbf{r})}{J(\mathbf{r},t)}$$

- The Lagrangian is: $L = T - V$

$$T = \int d^3u \frac{1}{2}\rho(\mathbf{r},t)v^2 = \int d^3r \rho_0(r)\frac{v^2}{2}$$
$$V = \int d^3u \rho\epsilon = \int d^3r \rho_0(r)\epsilon$$

- so the Lagrangian-density is $\Lambda = \rho_0 \left(\frac{v^2}{2} - \epsilon\right)$
- The Euler–Lagrange-equation in this formalism:

$$\frac{\partial\Lambda}{\partial u_a} = \frac{\partial}{\partial t}\frac{\partial\Lambda}{\partial u_{a,i}} + \frac{\partial}{\partial r_b}\frac{\partial\Lambda}{\partial u_{a,b}}$$

Classical hydrodynamics – Basic equation

- Let see some thermodynamics!
- On the trajectory, for the adiabatical process the entropy is constant
- The first principal of thermodynamics

$$d\epsilon = \frac{pd\rho}{\rho^2} + Tds$$

so on the trajectory: $d\epsilon = \frac{pd\rho}{\rho^2}$ Calculate the derivates:

$$\frac{\partial \Lambda}{\partial u_{a,b}} = \frac{\partial \Lambda}{\partial J_{ab}} = -\rho \frac{\partial}{J_{ab}} \epsilon \left(\frac{\rho_0}{J}, s_0 \right) = \frac{\rho_0^2}{J^2} \frac{\partial \epsilon}{\partial \rho} \frac{\partial J}{\partial J_{ab}} = pJJ_{ba}^{-1}$$

$$\frac{\partial \Lambda}{\partial u_a} = 0$$

$$\frac{\partial \Lambda}{\partial u_{a,t}} = \frac{\partial \Lambda}{\partial v_a} = \rho_0 v_a$$

Classical hydrodynamics – Basic equation

- Put everything together the equation of motion:

$$\rho_0 \frac{\partial v_a}{\partial t} = -\frac{\partial}{\partial r_b} pJ J^{-1}_{ba}$$

- Don't forget: this equation is calculated in Lagrange picture. The EMT is:

$$T_{ij} = u_{a,i} \frac{\partial \Lambda}{\partial u_{a,j}} - \Lambda \delta_{ij}$$

$$T_{00} = e = v_a \frac{\partial \Lambda}{\partial v_a} - \Lambda = \rho_0 \left(\frac{v^2}{2} + \epsilon \right)$$

$$T_{0a} = [\mathbf{j}_e]_a = v_a \frac{\partial \Lambda}{\partial u_{b,a}} = v_b pJ J^{-1}_{ab} = pJ[\mathbf{J}^{-1}\mathbf{v}]_a$$

$$T_{a0} = [\mathbf{p}]_a = u_{b,a} \frac{\partial \Lambda}{\partial v_a}$$

$$T_{ab} = u_{c,a} \frac{\partial \Lambda}{\partial u_{c,b}} - \Lambda \delta_{a,b} = \delta_{a,b}(pJ - \Lambda)$$

Classical hydrodynamics – Basic equation

- In the Euler picture, we get the well-known form:

$$\text{Continuity-equation}: \frac{\partial \rho}{\partial t} + \nabla \rho \mathbf{v} = 0$$

$$\text{Euler-equation}: \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p$$

$$\text{Energy-equation}: \frac{\partial \epsilon}{\partial t} + \nabla(\mathbf{v}\epsilon) = -\frac{p}{\rho}\nabla \mathbf{v}$$

- There are more unknown quantity than the number of the equations
- We have to give the Equation of State (EoS): $\epsilon = \kappa p$
- In the most of the case κ is a constant:

$$c_s = \sqrt{\frac{\partial p}{\partial \epsilon}} \rightarrow \frac{1}{c_s^2} = \frac{\epsilon}{p} = \kappa$$

Classical hydrodynamics – An example solution

Csörgő etal. arXiv:hep-ph/0108067v4

- If we know all these, we can solve the equation theoritically
- We interest in self-similar, ellipsoidal solution

$$n(t,\mathbf{r}) = n_0 \frac{V_0}{V} e^{-\frac{s}{2}}$$

$$\mathbf{v}(t,\mathbf{r}) = \left(\frac{\dot{X}}{X} r_x, \frac{\dot{Y}}{Y} r_y, \frac{\dot{Z}}{Z} r_z \right)$$

$$T(t,\mathbf{r}) = T_0 \left(\frac{V_0}{V} \right)^{\frac{1}{\kappa}}$$

- Where $s = -\frac{r_x^2}{2X^2} - \frac{r_y^2}{2Y^2} - \frac{r_z^2}{2Z^2}$ scale function
- It is only true if: $X\ddot{X} = Y\ddot{Y} = Z\ddot{Z} = \frac{T}{m}$

Importance of the relativistic case

- The previous solution is classical, 3+1 dimensional, ellipsoidal
- Great, but we want to do relativistic hydrodynamics
- Why? Later we will talk about some problem which are have to be investigated with relativistic hydrodynamics
- In the prevoius solution we saw there is an expanding fireball
- What if the velocity of the expanding is relativistic?
- What if the energy density is relativistic?

Relativistic hydrodynamics – Basic equation

- We can use field theory in this section too!
- If we get the follow Lagrangian density

$$\Lambda = -\rho \cdot \epsilon(\rho, s) + \frac{\rho\nu}{2}\left(1 - g^{ij}u^i u^j\right)$$

- ρ, ϵ, s are in the co-moving system
- From variational principals we can get the general Euler-equation:

$$\frac{Dwu_i}{d\tau} = (wu_i)_{;j}u^j = \frac{p_i}{\rho}$$

- where $w = \frac{\epsilon + p}{\rho}$
- From the Lagrangian-density we can get the canonical EMT:

$$T^{ij} = \phi_{\alpha,i}\frac{\partial\Lambda}{\partial\phi_{\alpha,i}} - g^{ij}\Lambda = \rho w u^i u^j - g^{ij}p$$

Relativistic hydrodynamics – Basic equation

- The energy conservation and the equation of motion can be calculated from the EMT with $\partial_\nu T^{\mu\nu} = 0$

$$\text{Energy equation} : (\epsilon + p)\partial_\nu u^\nu + u^\nu \partial_\nu \epsilon = 0$$
$$\text{Euler equation} : (\epsilon + p) u^\nu \partial_\nu u^\mu = (g^{\mu\nu} - u^\mu u^\nu)\partial_\nu p$$

- Of course, we have to write the continuity equation and the EoS:

$$\text{Continuity equation} : \partial_\mu (n u^\mu) = 0$$
$$\text{EoS} : \epsilon = \kappa(T) p$$

Why we do this?

- It is a good queation, why we care with relativistic hydrodynamics?
- Of course, it is nice, but it is not enough
- The L&K and the B&H solution were motivated by the heavy ion collisions
- We can describe these reaction phenomenologically
- There are just few exact solution, and we have no one which is accelerating, 3+1 dimensonial and relativistic with arbitrary geometric

Heavy Ion Collision

- Jet quenching – missing particle with large momentum (Phys.Rev.Lett. 88.022301 (2002)).
- The test against (Au+Au, d+Au) (Phys. Rev. Lett. 91, 072303 (2003))
- The fluid dynamics has right($v_2 \neq 0$) (Nucl. Phys. A 757, 184-283 (2005))
- Scale behavior (quark degrees of freedom appear) (Phys. Rev. Lett. 98, 162301 (2007))
- Almost perfect fluid (Phys. Rev. Lett. 98, 172301 (2007))
- Very high initial temperature (Phys. Rev. Lett. 104, 132301 (2010))
- Fluid containts free gluon and quark
- It exists for very long time: ($\tau_0 \approx 8\text{fm}/c \approx 10^{-23}s$)

Some well-known solution

- Landau–Khalatnikov-solution (L&K)
  (S. Belen'kji and L. Landau, Il Nuovo Cimento (1955-1965) 3, 15 10.1007/BF02745507 (1956).)
  1. 1+1 dimesion
  2. Implicit
  3. Longitudinal
  4. Can be applied to $p^+ - p^+$ collision

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- Hwa–Bjørken-solution (H&B)
  (R. C. Hwa, Phys. Rev. D 10, 2260 (1974). ; J. D. Bjorken, Phys. Rev. D 27, 140 (1983).)
  1. 1+1 dimension
  2. Accelerationless
  3. Explicit
  4. Underestimating the initial energy density

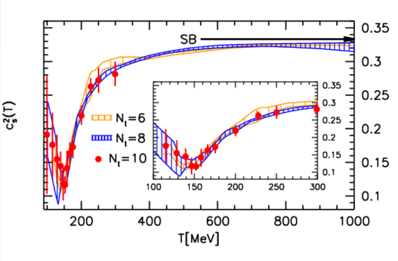
An example solution with more details

Csörgő, Csernai, Hama *et al.*, Heavy Ion Phys. **A21**, 73 (2004), nucl-th/0306004

- The velocityfield is Hubble-type:
  $u^\mu = \gamma\left(1, \frac{\dot{X}}{X}x, \frac{\dot{Y}}{Y}y, \frac{\dot{Z}}{Z}z\right) = \frac{x^\mu}{\tau}$
- $\dot{X}, \dot{Y}, \dot{Z} = const. \rightarrow$ accelerationless
- The thermodynamical quantities are:
  - $n = n_0 \left(\frac{\tau_0}{\tau}\right)^3 \nu(s)$
  - $T = T_0 \left(\frac{\tau_0}{\tau}\right)^{3/\kappa} \frac{1}{\nu(s)}$
  - $p = p_0 \left(\frac{\tau_0}{\tau}\right)^{3+3/\kappa}$
- n a barion-density, T the temperature, p the pressure, κ compressibility and τ the proper time.
- $\nu(s) = e^{-bs/2}$ arbitrary function
- $s = \frac{x^2}{X^2(t)} + \frac{y^2}{Y^2(t)} + \frac{z^2}{Z^2(t)}$ the scale parameter.

Realistic EoS – What's wrong?

- $\kappa = const.$ is not realistic: Katz et al. JHEP 1011, 077 (2010) [arXiv:1007.2580]



- In the interesting region the $\kappa \neq const.$
- Assume the $\kappa = \kappa(T)$
- If we can solve the basic equation of hydrodynamics with this EoS may we can describe this region

Realistic EoS – Conserved charge

Csanád, Nagy, Lökös, EPJ A (2012) **48**:173

- $\kappa = \kappa(T) \to \epsilon = \kappa(T)p$ generalization
- The velocity field is Hubble-type $u^\mu = \frac{x^\mu}{\tau}$ and the volume is $V = \tau^3$
- The conserved charge is $n = n_0 \cdot \left(\frac{V_0}{V}\right)$
- If we put these in the energy equation a temperature equation can be calculated:

$$u^\mu \left[\frac{\partial_\mu V}{V} + \frac{d(\kappa(T)T)}{dT} \frac{\partial_\mu T}{T} \right] = 0$$

- $\frac{\tau_0^3}{\tau^3} = \exp \int_{T_0}^{T} \frac{d(\kappa(T')T')}{dT'} \frac{\partial_\mu T'}{T'} dT'$

Realistic EoS – Non-conserved charge

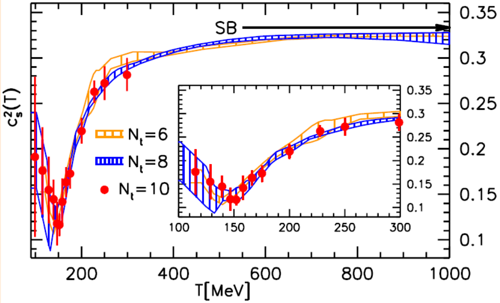
Csanád, Nagy, Lökös, EPJ A (2012) **48**:173

- If there are no any conserved charge we can use some thermodynamical equation:
  $\epsilon = \kappa(T)p$, $\epsilon = Ts - p \rightarrow d\epsilon = Tds$
- If we use $\epsilon + p = Ts, d\epsilon = Tds$ relations the entropy-conservation can be yield in a continuity equation
- If we use $\epsilon + p = Ts$ we can calculate a new solution
  $\frac{\tau_0^3}{\tau^3} = \exp \int_{T_0}^{T} \left(\frac{\kappa}{T'} + \frac{1}{\kappa+1} \frac{\kappa(T')}{dT'} \right) dT'$
- If we assume $\kappa = \kappa(p) \rightarrow$
  $\frac{\tau_0^3}{\tau^3} = \exp \int_{p_0}^{p} \left(\frac{\kappa}{(\kappa+1)p} + \frac{1}{\kappa+1} \frac{d\kappa}{dp} \right) dp$
- This solution can be transformed into the prevoius solution if $\kappa = \kappa(p(T))$

lQCD results

Borsányi, Fodor, Katz *et al.* JHEP **1011**, 077 (2010), arXiv:1007.2580

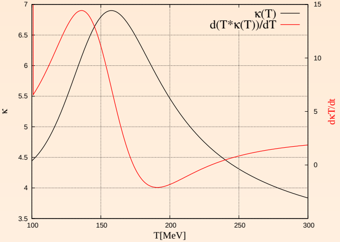
- Now, we have to get a $\kappa(T)$ function
- Fortunatly: lQCD parametrization is exist

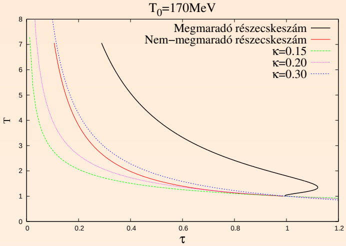


- From the parametrization of the trace anomaly, $I(T)/T^4$ we can get the pressure: $\frac{p(T)}{T^4} = \int_0^T \frac{dT}{T}\frac{I(T)}{T^4}$
- $I = \epsilon - 3p \rightarrow \kappa = \frac{I}{p} + 3$

Case of the conserved charge

- With conserved charge we get an unrealistic result: $\frac{d\kappa}{dT} < 0 \rightarrow \partial_\nu u^\nu \leq 0$

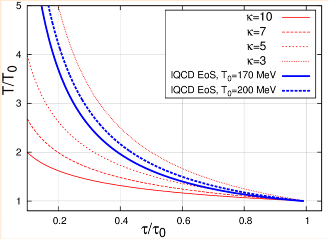




- The validity is **limited**, if $T_0 = 173 - 225\text{MeV}$

Case of the non-conserved charge

- The previous problem will not appear if there is no conserved charge



- Assume $\tau_{init} = 1.5\text{fm}$, $\tau_f = 8\text{fm}$ and the $T_0 = 170\text{MeV}$ (earlier hydrofits yields these parameters)
- Initial energy is a little higher than early calculation predicted it: $E_{init} = 550MeV$

Thank you for your attention!